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Inherent robustness properties of quasi-infinite horizon nonlinear model predictive control[☆]

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ABSTRACT

We consider inherent robustness properties of model predictive control (MPC) for continuous-time nonlinear systems with input constraints and terminal constraints. We show that MPC with a nominal prediction model and persistent but bounded disturbances has some degree of inherent robustness when the terminal control law and the terminal penalty matrix are chosen as the linear quadratic control law and the related Lyapunov matrix, respectively. We emphasize that the input constraint sets can be any compact set rather than convex sets, and our results do not depend on the continuity of the optimal cost function or of the control law in the interior of the feasible region.

1. Introduction

Model predictive control (MPC) has received remarkable attention in both practical applications and theoretical research over the last 30 years since it is capable of explicitly dealing with state and input constraints (Mayne, Rawlings, Rao, & Scokaert, 2000; Qin & Badgwell, 2003). The basic idea of standard MPC (Chen & Allgöwer, 1998; Fontes, 2001; Magni, De Nicolao, & Scattolini, 2001; Mayne et al., 2000) is as follows: Online, a finite horizon open-loop optimal control problem based on the current measurement of the system states is solved. Then, the first part of the obtained open-loop optimal input trajectory is applied to the system. At the succeeding time instant, the optimal control problem is solved again using new state measurements and with a shifted horizon, and the actual control input is updated.

For a nominally stabilizing model predictive control (MPC) scheme the presence of disturbances and/or model uncertainties may lead to performance deterioration or even loss of stability. An intuitive approach to guarantee robust stability and recursive feasibility is to use a min–max MPC formulation, where the optimal input is determined such that the performance criteria is minimized for the worst-case uncertainty (Bemporad, Borrelli, & Morari, 2003; Chen, Scherer, & Allgöwer, 1997; Fontes & Magni, 2003; Limon, Alamo, Salas, & Camacho, 2006; Magni, De Nicolao, Scattolini, & Allgöwer, 2003; Raimondo, Limon, Lazar, Magni, & Camacho, 2009; Scokaert & Mayne, 1998). However, such approaches are usually computationally expensive. Furthermore, the optimal input is obtained for a possibly unrealistic worst-case scenario, which often results in poor performance in the case of small actual uncertainties. Constraint tightening approaches, as introduced by Chisci, Rossiter, and Zappa (2001), Limon, Alamo, and Camacho (2002) and Richards and How (2006), can avoid computational complexity by using a nominal prediction model and tightened constraint sets. However, the constraint sets shrink drastically because the "margin", which reflects the effect of uncertainties, increases exponentially with the increase of the prediction horizon. For linear discrete-time systems with persistent disturbances, Mayne, Seron, and Rakovic (2005) and Rawlings and Mayne (2009) provide a new constraint tightening, tube based robust MPC scheme, which has fixed tightened sets. The results utilize both state feedback control and feedforward control action, and have been extended by Rakovic, Teel, Mayne, and Astolfi (2006); Yu,

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Böhm, Chen, and Allgöwer (2010) to systems with matched nonlinearity and piecewise affine systems. The tube based robust MPC scheme has also been extended to general discrete-time nonlinear systems (Mayne, Kerrigan, van Wyk, & Falugi, 2011). It possesses two loops, where a nominal MPC scheme in the inner loop generates a reference trajectory and the MPC control in the outer loop steers trajectories of the uncertain systems towards the reference trajectory. The schemes are based on the *a priori* estimation of the effect of disturbances over the prediction horizon.

Since robust MPC methods are much more complex than those developed for the nominal case, it is of interest to analyze under which conditions nominal MPC can guarantee robustness with respect to specific classes of disturbances. The paper (Grimm, Messina, Tuna, & Teel, 2004) used examples to illustrate that MPC applied to nonlinear systems can produce nominal asymptotic stability without any robustness, when the optimization problem contains state constraints or equality terminal constraints. In the examples, either the considered nonlinear system is discontinuous at its equilibrium or the Jacobian linearization of the considered nonlinear systems is not stabilizable. Under the fundamental assumption that the presence of uncertainties or disturbances do not cause any loss of feasibility, robustness properties of nominal MPC algorithms are proved in Magni and Sepulchre (1997), Nicolao, Magni, and Scattolini (1996) and Scokaert, Rawlings, and Meadows (1997). The recursive feasibility assumption holds true when the problem formulation does not include state and input constraints and when the terminal constraint used to guarantee nominal stability is also satisfied under perturbed conditions (Magni & Scattolini, 2007). For unconstrained input-affine nonlinear systems, it is shown in Magni and Sepulchre (1997) that the nominal MPC control law is inverse optimal. Thus, it is also optimal for a modified optimal control problem spanning over an infinite horizon. Due to this inverse optimality property, the MPC control law inherits the same robustness properties as the infinite horizon optimal control assuming that the sampling time goes to zero. Under the assumption that the optimal cost function is twice continuously differentiable, it has been shown in Nicolao et al. (1996) that MPC control law provides robustness with respect to gain perturbations due to actuator and additive perturbations describing unmodeled dynamics. Results on inherent robustness with exponentially decaying disturbances are reported in Scokaert et al. (1997) with the assumption that the MPC control law is Lipschitz continuous. The papers (Findeisen & Allgöwer, 2005; Limon et al., 2009; Pannocchia, Rawlings, & Wright, 2011) show that nominal MPC possesses inherent robustness properties if the optimal cost function is locally Lipschitz continuous or the MPC control law is regionally continuous. However, both the resulting MPC control law and the optimal value function associated to the optimization problem defining nominal MPC can be discontinuous (Fontes, 2000; Meadows, Henson, Eaton, & Rawlings, 1995; Rawlings & Mayne, 2009). While MPC is applied to linear systems with convex constraints, some robustness exists (Grimm et al., 2004). The result depends on the fact that continuity of the optimal value function on the interior of the feasible region is a sufficient condition for robustness, as is continuity of the feedback law on the interior of the feasible region (Jiang & Wang, 2001). The paper (Grimm, Messina, Tuna, & Teel, 2007) shows that the system under control is robust to sufficient small disturbances, if (a) the value function is bounded by a $\mathcal{K}_\infty$ function of a state measure (related to the distance from the state to some target set) and this measure is detectable from the stage cost used in the MPC algorithm; (b) the systems satisfy a definition that attempts to characterize the robustness properties of the MPC optimization problem. Instead of the analysis of the inherent robustness properties of existing nominal MPC schemes, Lazar and Heemels (2009) and Picasso, Desiderio, and Scattolini (2010, 2011, 2012) propose novel nominal MPC schemes which have some inherent robustness properties.

Quasi-infinite horizon MPC (Chen & Allgöwer, 1998; Mayne et al., 2000) is one of the main results of nonlinear MPC with guaranteed nominal stability. Our previous conference paper (Yu, Reble, Chen, & Allgöwer, 2011) considers the inherent robustness properties of quasi-infinite horizon MPC with input constraints and a terminal constraint. Although the recursive feasibility is proved directly, the proof of robust stability is not complete. In this paper, we rigorously show inherent robustness properties of quasi-infinite horizon MPC of nonlinear systems with input constraints, where the disturbances are persistent but bounded and the optimization problem has a terminal constraint. It is worth noting that the following analysis does neither assume the continuity of the optimal cost function nor of the control law, and hence the results are more general than previous results available in the literature. It is shown that the degree of robustness depends on the terminal set and on the terminal penalty function, the prediction horizon, the upper bound on the disturbances, and the Lipschitz constant of the system.

The remainder of the paper is organized as follows. The problem is set up in Section 2. Terminal conditions for nominal stability, recursive feasibility of the online optimization problem, and robust stability are proposed in Section 3. Further results on inherent robustness properties of linear MPC is discussed in 4. Section 5 provides two examples to demonstrate the effectiveness of the derived results.

1.1. Notations and basic definitions

Let $\mathbb{R}$ denote the field of real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{Z}_{[0,\infty)}$ the field of non-negative integers. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the 2-norm and $\|v\|_Q = \sqrt{v^T Q v}$ with $Q \in \mathbb{R}^{n\times n}$ and $Q > 0$. Let $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of the eigenvalues of matrix M and $\sigma(M)$ the largest singular value of M. The operation $\oplus$ is the addition of sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, $\mathcal{A} \oplus \mathcal{B} := \{a + b \in \mathbb{R}^n | a \in \mathcal{A}, b \in \mathcal{B}\}$. The operation $\ominus$ is the subtraction of sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, where $\mathcal{A} \ominus \mathcal{B} := \{x \in \mathbb{R}^{n_x} | \{x\} \oplus \mathcal{B} \subseteq \mathcal{A}\}$. Denote the set $\mathcal{B}(x_0, \delta) := \{x \in \mathbb{R}^n \mid \|x - x_0\| \le \delta\}$, $\mathcal{B}(\delta) := \{x \in \mathbb{R}^n \mid \|x\| \le \delta\}$, and $\varnothing$ as the empty set. Denote $\mathbb{L}^n_{[a,b]}$ as the space of all Lebesgue functions mapping from $[a, b]$ to $\mathbb{R}^n$.

We introduce the following definitions which will be used in the paper:

Definition 1. A system is ultimately bounded if the system converges asymptotically to a bounded set.

Definition 2 (*Hausdorff Distance Rawlings & Mayne, 2009*)**.** The Hausdorff distance $d(\cdot, \cdot)$ between two sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$ is defined by

$$d(\mathcal{X}, \mathcal{Y}) := \max\left\{\sup_{x\in\mathcal{X}} d(x, \mathcal{Y}),\ \sup_{y\in\mathcal{Y}} d(y, \mathcal{X})\right\},$$

in which $d(a, \mathcal{M})$ denotes the distance of a point $a \in \mathbb{R}^n$ from a set $\mathcal{M} \subset \mathbb{R}^n$, which is defined by

$$d(a, \mathcal{M}) := \inf_{b\in\mathcal{M}} d(a, b),$$

where $d(a, b) = \|a - b\|$.

Definition 3 (*Relation*)**.** Suppose that both $\mathcal{X}$ and $\mathcal{Y}$ are compact sets with $\mathcal{X} \subseteq \mathcal{Y} \subset \mathbb{R}^n$, and $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ are the boundaries of sets $\mathcal{X}$ and $\mathcal{Y}$, respectively. The relation $d_r(\cdot, \cdot)$ between sets $\mathcal{X}$ and $\mathcal{Y}$ is defined by

$$d_r(\mathcal{X}, \mathcal{Y}) := \min_{x\in\bar{\mathcal{X}},\, y\in\bar{\mathcal{Y}}} \|x - y\|.$$

2. Problem setup

Consider nonlinear continuous-time systems with additive exogenous disturbances

$$\dot{x}_R(t) = f(x_R(t), u(t)) + w(t), \tag{1a}$$

$$w(t) \in \mathcal{W}, \tag{1b}$$

where $x_R(t) \in \mathbb{R}^{n_x}$ denotes the system state and $u(t) \in \mathbb{R}^{n_u}$ the control input at time instant t, and $w(t) \in \mathcal{W} \subset \mathbb{R}^{n_x}$ represents a persistent disturbance. Here, we assume that $\mathcal{W} := \{w \in \mathbb{R}^{n_x} \mid \|w\| \leq \beta\}$, i.e., the norm of the disturbance is bounded.

The system is subject to the control constraint

$$u(t) \in \mathcal{U}, \tag{2}$$

where the set $\mathcal{U} \subset \mathbb{R}^{n_u}$ is a compact set and contains $0 \in \mathbb{R}^{n_u}$ in its interior.

The nominal dynamics of the system (1) is defined by

$$\dot{x}(t) = f(x(t), u(t)). \tag{3}$$

The optimization problem in the quasi-infinite horizon MPC is formulated as follows:

Problem 4.

$$\begin{aligned}
&\underset{u(\cdot)}{\text{minimize}}\ J(x(t), u(\cdot)) \\
\text{subject to}\quad &\dot{x}(\tau, x(t)) = f(x(\tau, x(t)), u(\tau)), \\
&x(t, x(t)) = x(t), \\
&u(\tau) \in \mathcal{U}, \quad \tau \in [t, t+T_p], \\
&x(t+T_p, x(t)) \in \mathcal{X}_f,
\end{aligned}$$

where

$$J(x(t), u(\cdot)) := \|x(t+T_p, x(t))\|_P^2 + \int_t^{t+T_p} \left(\|x(s, x(t))\|_Q^2 + \|u(s)\|_R^2\right) ds,$$

is the cost functional, T_p is the prediction horizon, $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive definite state and input weighting matrices. The positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ is the terminal penalty matrix, and $E(x) := \|x\|_P^2$ is the terminal penalty function. The terminal set $\mathcal{X}_f := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha\}$ is a level set of the terminal penalty function. The term $x(\cdot, x(t))$ represents the predicted state trajectory starting from the initial state $x(t)$ under the control $u(\cdot)$. For simplicity, denote the optimal value of $J(x(t), u(\cdot))$ as $J^0(x(t))$. In order to guarantee feasibility and nominal stability, P and $\mathcal{X}_f$ have to satisfy terminal conditions, see Chen and Allgöwer (1998) and Mayne et al. (2000). We will introduce these conditions in Section 3.

The goal of this paper is to determine the largest upper bound β on the disturbance w such that the real system is robustly stable for all $w \in \mathcal{W}$, i.e., the real system under *nominal* MPC control law is inherently robust, and can endure persistent disturbances which are less than β in the sense of the norm. Notice here that in Problem 4, the *nominal* system is used as prediction model and no disturbances are taken into account.

Some standing assumptions are stated in the following:

Assumption 5. The system state x can be measured instantaneously.

Assumption 6. f is twice continuously differentiable, and $f(0, 0) = 0$. Thus, $0 \in \mathbb{R}^{n_x}$ is an equilibrium of the nominal system.

According to the principle of MPC, the optimization problem will be solved repeatedly, when new measurements are available at the sampling instants $t_j = j\delta$, where δ is a sampling time and $0 < \delta \leq T_p$, $j \in \mathbb{Z}_{[0,\infty)}$.

Assuming that the minimum is attained, the optimal solution to Problem 4 is given by the optimal input trajectory, and for any $\tau \in [t, t+T_p]$

$$u^*(\tau, x(t)) := \arg \min_{\substack{u(\cdot)\in\mathcal{U} \\ x(t+T_p, x(t))\in\mathcal{X}_f}} J(x(t), u(\tau)).$$

The applied control is $u^*(\tau, x(t))$, $\tau \in [t, t+\delta)$.

3. Inherent robustness to persistent disturbances

As pointed out before, robust MPC usually leads to either heavy computational burden or poor performance, or both of them. In this section, we discuss the inherent robustness properties of nominal MPC rather than propose a new robust MPC scheme. First, we recall a common way to construct the terminal set and the terminal penalty function of quasi-infinite horizon MPC, which will play an important role in the analysis of the robustness properties.

3.1. Terminal conditions for nominal stability

Lemma 7 (Chen & Allgöwer, 1998)**.** *Suppose that the Jacobian linearization of the nominal system at the origin is stabilizable, $K \in \mathbb{R}^{n_u \times n_x}$ is the linear quadratic regulator (LQR) optimal feedback matrix of the linearized system with weighting matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$, where $Q > 0$ and $R > 0$. Then, the Lyapunov equation*

$$(A + BK + \kappa I_{n_x})^T P + P(A + BK + \kappa I_{n_x}) = -Q^*,$$

where $Q^* = Q + K^T RK$ is positive definite, admits a unique positive definite and symmetric solution $P \in \mathbb{R}^{n_x \times n_x}$, whenever $\kappa \in [0, -\lambda_{\max}(A+BK))$. Furthermore, there exists a constant $\alpha \in (0, \infty)$ specifying a neighborhood $\mathcal{X}_f := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha\}$ of the origin such that

(1) *$Kx \in \mathcal{U}$, for all $x \in \mathcal{X}_f$, i.e., the linear feedback controller respects the input constraints in $\mathcal{X}_f$,*
(2) *$\mathcal{X}_f$ is positively invariant for the nominal system controlled by the local linear feedback $u = Kx$.*

Notice that $u(\tau) = Kx(\tau)$, $\tau \in [t, t+T_p]$, renders $\mathcal{X}_f$ positively invariant. Hence, it is a feasible solution to Problem 4 provided that $x(t) \in \mathcal{X}_f$.

Remark 8. The terminal control law Kx is never actually applied to the system in quasi-infinite horizon MPC, and only used to compute the terminal penalty function and the terminal set, see also Chen and Allgöwer (1998).

In the set $\mathcal{X}_f$, $u = Kx$ guarantees that $\frac{dE(x)}{dt} \leq -x^T Q^* x$ along the nominal trajectory, where Q^* is defined in Lemma 7. Then, there exists a positive constant π such that

$$-x^T Q^* x \leq -\pi x^T P x, \quad \forall x \in \mathcal{X}_f,$$

i.e., the linear control law $u = Kx$ renders the nominal system exponentially stable in $\mathcal{X}_f$. The decay rate π can be chosen as $\pi \leq \pi_0 = \lambda_{\min}(Q^*)/\lambda_{\max}(P)$. This will help us understand the behavior of the system dynamics under the terminal control law. Furthermore, we will prove in Section 3.3 that the MPC control law has the same robustness properties as the terminal control law.

Lemma 9. *If the state $x(t)$ of the nominal system (3) lies in $\mathcal{X}_f$ at time instant t, then there exists an $s \in [t, t+\delta]$ such that the system* *trajectory under the terminal control law enters into the set*

$$\Omega := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \le e^{-\pi_0 \delta}\alpha\right\}$$

at the time instant s.

Proof. Since $x(t) \in \mathcal{X}_f, x(t)^T P x(t) \le \alpha$. Due to $\frac{dE(x)}{dt} \le -x^T Q^* x$ and $-x^T Q^* x \le -\pi_0 x^T P x$, we have $\frac{dE(x)}{dt} \le -\pi_0 x^T P x$. Therefore, $x(t+\delta)^T P x(t+\delta) \le e^{-\pi_0\delta}\alpha$.

Notice that $\Omega \subset \mathcal{X}_f$ since $e^{-\pi_0\delta} < 1$. If the considered optimization problem is feasible at the initial time instant and if there is no disturbance or model uncertainty, then the optimization problem is feasible for all time instants t and $J^0(x(t_\pi)) - J^0(x(t_\upsilon)) \le -\int_{t_\upsilon}^{t_\pi}\left(\|x(s)\|_Q^2 + \|u(s)\|_R^2\right) ds$, where t_π and t_υ are any two successive sampling instants (Chen & Allgöwer, 1998). If the actual system trajectory deviates sufficiently small from the nominal system trajectory, we can still have $J^0(x(t_\pi)) - J^0(x(t_\upsilon)) < 0$. In other words, the optimal cost function $J^0(x)$ is a candidate Lyapunov function and provides some degree of robustness if $J^0(x)$ is continuous in x. However, it is well-known that recursive feasibility is not ensured by the nominal cost constrained minimization for future states under any disturbance realization (Chisci et al., 2001), and $J^0(x)$ is not necessarily continuous in x (Grimm et al., 2004). In the following, an upper bound β on the disturbance w is estimated which will preserve the guaranteed recursive feasibility of Problem 4 if the online optimization problem is feasible at the initial time instant. Then, robust stability is addressed by showing that the system trajectory will converge to a set around the origin.

3.2. Robust recursive feasibility

In this subsection, we introduce a general lemma which provides a useful estimate on the deviation of the real system state from the nominal system state. Based on this lemma, we discuss the recursive feasibility of nominal MPC with respect to persistent but bounded disturbances.

Let r be a given constant. The set $\mathcal{B}(r)$ is convex and compact. Since $f(\cdot,\cdot)$ is twice continuously differentiable on $\mathcal{B}(r) \times \mathcal{U}$, and $\mathcal{U}$ is a compact set, $\|\partial f/\partial x\|$ is bounded on $\mathcal{B}(r) \times \mathcal{U}$. Hence, f is Lipschitz continuous with respect to x with some Lipschitz constant $\upsilon \ge 0$

$$\|f(y,u) - f(x,u)\| \le \upsilon\|y - x\|. \tag{4}$$

This allows to use the following lemma.

Lemma 10 (*Khalil, 2002*)**.** *Consider the real system (1) and the nominal system (3), where $f(\cdot,\cdot)$ is a continuously differentiable function. Suppose that $\|w(t)\| \le \beta$, then the norm of $\|x_R(t) - x(t)\|$ satisfies the following inequality*

$$\|x_R(t) - x(t)\| \le \|x_R(0) - x(0)\|e^{\upsilon t} + \frac{\beta}{\upsilon}(e^{\upsilon t} - 1), \quad \forall t \in [0, T_p + \delta], \tag{5}$$

where $x(\theta) \in \mathcal{B}(r)$ and $x_R(\theta) \in \mathcal{B}(r)$ for all $\theta \in [0, t]$.

The following lemma states that if the disturbance is small enough, the real system trajectory will stay in a tube along the nominal trajectory during the interval $t \in [0, T_p + \delta]$.

Lemma 11. *Let $x(t)$ be the solution of the nominal system (3) with $u(t) \in \mathcal{U}$ for all $t \in [0, T_p + \delta]$, and $x(0) = x_0$. Given $\epsilon > 0$, the trajectory $x_R(\cdot)$ of the real system (1) defined on $[0, T_p + \delta]$, with $x_R(0) = x_0$ and $u(t)$, lies in the tube*

$$S(x_0, \epsilon) := \left\{(t, x_R) \in [0, T_p + \delta] \times \mathbb{R}^{n_x} \mid \|x_R - x(t)\| \le \epsilon\right\}$$

for all $\beta \in \left[0, \frac{\epsilon\upsilon}{e^{\upsilon(T_p+\delta)}-1}\right]$.

Proof. By continuity of $x(t)$ in t and the compactness of $[0, T_p + \delta]$, altogether with Assumption 6, we know that $x(t)$ is bounded on $[0, T_p + \delta]$. Furthermore, the set $S(x_0, \epsilon)$ is a compact set which contains $(t, x(t))$ for all $t \in [0, T_p + \delta]$. Because of (5) and $x_R(0) = x(0)$, we have

$$\|x_R(t) - x(t)\| \le \frac{\beta}{\upsilon}\left(e^{\upsilon t} - 1\right), \quad \forall t \in [0, T_p + \delta].$$

Since $\frac{\beta}{\upsilon}\left(e^{\upsilon t} - 1\right)$ is monotonically increasing in t for fixed υ, if β is small enough such that $\frac{\beta}{\upsilon}(e^{\upsilon(T_p+\delta)} - 1) \le \epsilon$, then $(t, x_R(t)) \in S(x_0, \epsilon)$ for all $t \in [0, T_p + \delta]$.

Denote the boundary of the set $\mathcal{X}_f$ as $\bar{\mathcal{X}}_f$, $\bar{\mathcal{X}}_f := \{x \in \mathbb{R}^{n_x} \mid x^T P x = \alpha\}$, and the boundary of the set Ω as $\bar{\Omega}$, $\bar{\Omega} := \{x \in \mathbb{R}^{n_x} \mid x^T P x = e^{-\pi\delta}\alpha\}$. The following lemma gives an estimate on the relation of the sets $\mathcal{X}_f$ and Ω.

Lemma 12. *The relation $d_r(\mathcal{X}_f, \Omega)$ of the sets $\mathcal{X}_f$ and Ω satisfies*

$$d_r(\mathcal{X}_f, \Omega) \ge \sqrt{\lambda_{\min}(P^{-1})} \cdot \left(\alpha^{\frac{1}{2}} - \left(e^{-\pi\delta}\alpha\right)^{\frac{1}{2}}\right).$$

Proof. Denote $h := P^{\frac{1}{2}}x$ and $s := P^{\frac{1}{2}}y$ and define $\tilde{\mathcal{X}}_f = \{h \in \mathbb{R}^{n_x} \mid h^T h = \alpha\}$ and $\tilde{\Omega} = \{s \in \mathbb{R}^{n_x} \mid s^T s = e^{-\pi\delta}\alpha\}$. Notice that $P^{\frac{1}{2}}$ is invertible since P is positive definite. Then,

$$\begin{aligned}\left[d_r(\mathcal{X}_f, \Omega)\right]^2 &= \min_{x \in \bar{\mathcal{X}}_f,\ y \in \bar{\Omega}} (x - y)^T(x - y) \\ &= \min_{h \in \tilde{\mathcal{X}}_f,\ s \in \tilde{\Omega}} (h - s)^T P^{-1}(h - s) \\ &\ge \lambda_{\min}(P^{-1}) \cdot \|h - s\|^2.\end{aligned}$$

Due to the triangle inequality $\|h - s\| \ge \|h\| - \|s\|$ holds. Moreover, $\|h\| = \alpha^{\frac{1}{2}}$ and $\|s\| = \left(e^{-\pi\delta}\alpha\right)^{\frac{1}{2}}$ for all $x \in \bar{\mathcal{X}}_f$ and $y \in \bar{\Omega}$, respectively. Hence,

$$\begin{aligned}d_r(\mathcal{X}_f, \Omega) &\ge \sqrt{\lambda_{\min}(P^{-1})} \cdot \|h - s\| \\ &\ge \sqrt{\lambda_{\min}(P^{-1})} \cdot (\|h\| - \|s\|) \\ &= \sqrt{\lambda_{\min}(P^{-1})} \cdot \left(\alpha^{\frac{1}{2}} - \left(e^{-\pi\delta}\alpha\right)^{\frac{1}{2}}\right).\end{aligned}$$

Denote $d_r^0(\mathcal{X}_f, \Omega) := \sqrt{\lambda_{\min}(P^{-1})}\left(\alpha^{\frac{1}{2}} - \left(e^{-\pi\delta}\alpha\right)^{\frac{1}{2}}\right)$ and

$$\beta_0 := \frac{\upsilon \cdot d_r^0(\mathcal{X}_f, \Omega)}{e^{\upsilon(T_p+\delta)} - 1}.$$

Suppose that β_0 is an upper bound on the disturbances, i.e., $\beta \le \beta_0$, then according to Lemma 11 the system trajectory of the real system (1) lies in the tube $S(x_0, \epsilon_0)$ along the nominal system trajectory in the interval $[0, T_p + \delta]$, where $\epsilon_0 := d_r^0(\mathcal{X}_f, \Omega)$.

For all $\epsilon \in (0, d_s^0(\mathcal{X}_f, \Omega)]$, if the terminal state of the nominal system is in the set Ω, then also the terminal state of the real system will be in the terminal set. That is, $x_R(T_p + \delta) \in \mathcal{X}_f$, see Fig. 1.

Now we are in a position to state the main result of this subsection, which shows the recursive feasibility of nominal MPC in the presence of disturbances.

Theorem 13. *Assume that Problem 4 has a feasible solution at $x(t_0)$, and denote the corresponding predicted nominal control and state as $u(\tau, x(t_0))$ and $x(\tau, x(t_0))$, respectively, $\tau \in [t_0, t_0 + T_p]$. Suppose that $\beta \le \beta_0$, then, $\tilde{u}_{t_0+\delta}(\cdot) \in \mathbb{L}^{n_u}_{[t_0+\delta, t_0+\delta+T_p]}$ with*

$$\tilde{u}_{t_0+\delta}(\tau) = \begin{cases} u(\tau, x(t_0)) & \tau \in [t_0 + \delta, t_0 + T_p], \\ Kx(\tau, x(t_0)) & \tau \in (t_0 + T_p, t_0 + \delta + T_p], \end{cases}$$

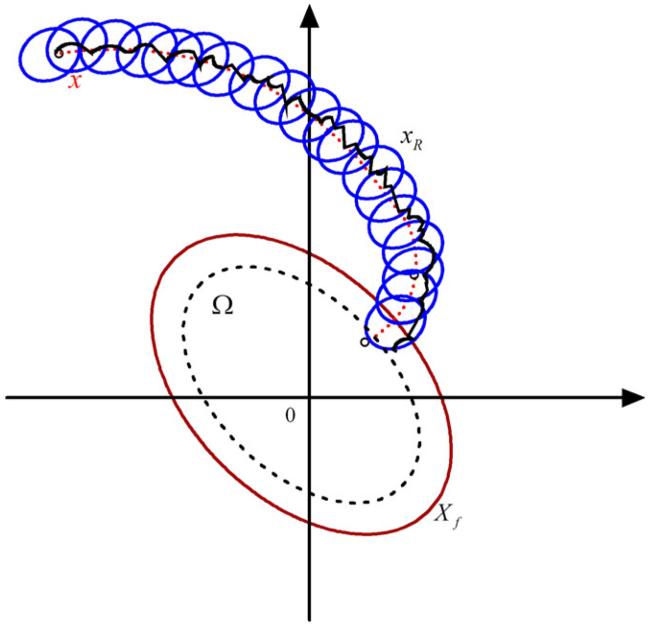

Fig. 1. Tube along the nominal trajectory in $[0, T_p + \delta]$. Dashed line shows the nominal trajectory, solid line is the real trajectory. Dashed ellipsoid depicts Ω, the larger ellipsoid represents $\mathcal{X}_f$.

is a feasible solution to Problem 4 at $x_R(t_0+\delta)$, where $x_R(t_0+\delta)$ is a point on the trajectory of the real system starting from $x(t_0)$ under the control $u(\tau, x(t_0))$, $\tau \in [t_0, t_0+\delta]$. The term $x(\tau, x(t_0))$, for all $\tau \in [t_0+T_p, t_0+T_p+\delta]$, is the nominal state trajectory starting from the state $x(t_0+T_p, x(t_0))$ under the linear control law Kx. Furthermore, $(t_0+\delta, x_R(t_0+\delta)) \in S(x(t_0), \epsilon_0)$.

Proof. Following Lemma 9, the control function $\bar{u}_{t_0}(\cdot) \in \mathbb{L}^{n_u}_{[t_0, t_0+\delta+T_p]}$ with

$$\bar{u}_{t_0}(\tau) := \begin{cases} u(\tau, x(t_0)) & \tau \in [t_0, t_0+T_p], \\ Kx(\tau, x(t_0)) & \tau \in (t_0+T_p, t_0+\delta+T_p], \end{cases}$$

drives the trajectory of the nominal system from $x(t_0)$ into the set Ω in the interval $[t_0, t_0+\delta+T_p]$, i.e., $x(t_0+\delta+T_p, x(t_0)) \in \Omega$. Moreover, the trajectory of the real system with the same control function will lie in the tube $S(x_0, \epsilon_0)$ for all $t \in [t_0, t_0+T_p+\delta]$ and for all $w \in \mathcal{W}$. Thus, $x_R(t_0+\delta+T_p, x(t_0)) \in \mathcal{X}_f$. Consequently, $\tilde{u}_{t_0+\delta}(\cdot)$ is a feasible solution to Problem 4 at $x_R(t_0+\delta)$.

Theorem 13 proves recursive feasibility, i.e., it is shown that there exists a feasible solution to Problem 4 at each time instant even in the presence of small bounded additive disturbances.

3.3. Robust stability

We consider system (1) with respect to persistent disturbances and the repeated application of open-loop inputs obtained as solutions to Problem 4. Due to the existence of persistent bounded disturbances, asymptotic stability might not be achieved. As a consequence, we desire only "ultimate boundedness" results.

Since the optimal cost function $J^0(x)$, which is chosen as the candidate Lyapunov function in the proof of nominal stability of quasi-infinite horizon MPC, is not necessarily continuous in the feasible region, it is hard to show inherent robustness properties of quasi-infinite horizon MPC directly. We prove that the system trajectory will converge to a set around the origin even though there exist persistent disturbances.

Assume that Problem 4 has an optimal solution at $x(t_i)$, and denote the corresponding predicted nominal control and state as $u^*(\tau, x(t_i))$ and $x^*(\tau, x(t_i))$, respectively, for $\tau \in [t_i, t_i+T_p]$.

Denote a function $\hat{u}_\phi \in \mathbb{L}^{n_u}_{[0,T_p]}$ with

$$\hat{u}_\phi(s) := \begin{cases} u^*(s+\phi, x(t_i)) & s \in [0, t_i+T_p-\phi), \\ Kx(s+\phi, x(t_i)) & s \in [t_i+T_p-\phi, T_p], \end{cases}$$

with parameter $\phi \in [t_i, t_i+T_p]$, and in which $x(s+\phi, x(t_i))$ denotes for all $s \in [t_i+T_p-\phi, T_p]$ the nominal state trajectory starting from the state $x^*(t_i+T_p, x(t_i))$ under the linear control law Kx.

Denote the set

$$H(x(t_i)) := \Big\{(t,x) \in [t_i, t_i+T_p] \times \mathbb{R}^{n_x} \mid \|x - x^*(t, x(t_i))\| \leq \frac{\beta}{v}\left(e^{v(t-t_i)} - 1\right)\Big\}, \tag{6}$$

which contains all states that can be reached along trajectories of the real system starting from $x(t_i)$, under the control $\hat{u}_{t_i}$ for all $w \in \mathcal{W}$ and for all $t \in [t_i, t_i+T_p]$, see Lemma 10. Furthermore, the set $H(x(t_i))$ is a compact set.

On account of Theorem 13, $\hat{u}_{t_i}(s)$ is the optimal solution to Problem 4 at $x(t_i)$ and $\hat{u}_\phi$ is a feasible (but not necessarily optimal) solution to Problem 4 at any initial state $x(\phi) := x_R(\phi, x(t_i))$ with $\phi \in [0, T_p]$.

Denote a set

$$\Psi(x(t_i)) := \left\{x \in \mathbb{R}^{n_x} \mid \|x - x^*(t_i+\delta, x(t_i))\| \leq \frac{\beta_0}{v}(e^{v\delta} - 1)\right\}, \tag{7}$$

which contains all possible states of the system (1) at the time instant $t_i+\delta$ starting from the state $x(t_i)$ under the control $\hat{u}_{t_i}$. For all $x \in \Psi(x(t_i))$, $\hat{u}_{t_i+\delta}$ is a feasible solution to Problem 4. For simplicity, denote the value of $J(x, \hat{u}_{t_i+\delta}(\cdot))$ as $\bar{J}(x)$. The next lemma shows that $\bar{J}(x)$ is continuous in $\Psi(x(t_i))$.

Lemma 14. *The cost function $\bar{J}(x)$ is continuous with respect to x for all $x \in \Psi(x(t_i))$.*

Proof. The proof consists of two parts. In the first part, we prove that the solution of

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{u}_{t_i+\delta}(t)), \qquad \hat{x}(0) = x \tag{8}$$

depends continuously on x for all $x \in \Psi(x(t_i))$. In the second part, we will prove the continuity of $\bar{J}(x)$ with respect to x.

First part: Let $\hat{x}(\cdot, x_1)$ and $\hat{x}(\cdot, x_2)$ be the solution of (8) starting from $x = x_1$ and $x = x_2$, respectively. Similar to Lemma 10, we have

$$\|\hat{x}(\tau, x_1) - \hat{x}(\tau, x_2)\| \leq \|x_1 - x_2\| e^{v\tau}, \quad \forall \tau \in [0, T_p].$$

For any $\varepsilon > 0$, we can choose $\delta = \varepsilon e^{-vT_p}$. Then for all x_1, x_2 which satisfy $\|x_1 - x_2\| \leq \delta = \varepsilon e^{-vT_p}$

$$\|\hat{x}(\tau, x_1) - \hat{x}(\tau, x_2)\| \leq \varepsilon$$

holds for all $\tau \in [0, T_p]$.

Second part: We emphasize that $\bar{J}(x) = J(x, \hat{u}_{t_i+\delta}(\cdot))$ is the cost function of the nominal system (8) along the prediction horizon. The continuity of $\bar{J}(x)$ follows from the continuity of the solution of (8) with respect to x and continuity of the functions $\|x\|_Q^2$ and $\|x\|_P^2$.

Since $\bar{J}(x) \geq 0$ for all $x \in \Psi(x(t_i))$, it has a lower bound; since both $\Psi(x(t_i))$ and $\mathcal{U}$ are compact sets, $\bar{J}(x)$ has an upper bound for all $x \in \Psi(x(t_i))$. Denote the supremum and the infimum of $\bar{J}(x)$ for all $x \in \Psi(x(t_i))$ by $M(x(t_i))$ and $m(x(t_i))$, respectively and define

$$\Delta M(x(t_i)) := M(x(t_i)) - m(x(t_i)).$$

Furthermore, due to the definition (7) $\Psi(x(t_i)) \to \{x^*(t_i+\delta, x(t_i))\}$ for $\beta \to 0$, i.e., the set Ψ converges to a single point. Hence, the supremum and infimum in this set coincide and, consequently, $\Delta M(x(t_i)) \to 0$ for $\beta \to 0$. Moreover, the convergence is uniform for all $x(t_i)$, see (7).

For the fixed prediction horizon T_p and the terminal set $\mathcal{X}_f$, Problem 4 is feasible at all time instants $t \geq t_0$ for system (1) with respect to persistent but bounded disturbances, if it is feasible at time instant $t = 0$. Denote the feasible set of system (1) as $\mathcal{X}_r$,

$$\mathcal{X}_r := \Big\{x_0 \in \mathbb{R}^{n_x} \mid \text{Problem 4 has a feasible solution for } x(0) = x_0\Big\}.$$

In order to show the boundedness of $\mathcal{X}_r$, we will introduce a lemma which is a variant of Gronwall–Bellman Inequality, see Khalil (2002, Lemma A.1).

Lemma 15. *Let $\lambda_h : [a, b] \to \mathbb{R}$ and $\mu_h : [a, b] \to \mathbb{R}$ be continuous and nonnegative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies*

$$y(t) \leq \lambda_h(t) - \int_b^t \mu_h(s)y(s)ds$$

for $a \leq t \leq b$, then in the same interval

$$y(t) \leq \lambda_h(t) + \int_t^b \lambda_h(s)\mu_h(s) \exp\left[-\int_s^t \mu_h(\tau)d\tau\right] ds.$$

Proof. The proof follows closely that of Khalil (2002, Lemma A.1). However, note that the sign of the integral is switched. Hence, Khalil (2002, Lemma A.1) is not directly applicable. Let $z(t) = \int_t^b \mu_h(s)y(s)ds$, and $v(t) = z(t) + \lambda_h(t) - y(t) \geq 0$. Then, z is differentiable and

$$\dot{z} = -\mu_h(t)y(t) = -\mu_h(t)[z(t) + \lambda_h(t) - v(t)].$$

This is a scalar linear state equation with the transition function

$$\Phi(t, s) = \exp\left[-\int_s^t \mu_h(\tau)d\tau\right].$$

Since $z(b) = 0$, we have

$$z(t) = \int_t^b \Phi(t, s)[\mu_h(s)\lambda_h(s) - \mu_h(s)v(s)]ds.$$

The term

$$\int_t^b \Phi(t, s)\mu_h(s)v(s)ds$$

is nonnegative. Therefore,

$$z(t) \leq \int_t^b \Phi(t, s)\mu_h(s)\lambda_h(s)ds = \int_t^b e^{-\int_s^t \mu_h(\tau)d\tau}\mu_h(s)\lambda_h(s)ds.$$

Since $y(t) \leq \lambda_h(t) + z(t)$,

$$y(t) \leq \lambda_h(t) + \int_t^b e^{-\int_s^t \mu_h(\tau)d\tau}\mu_h(s)\lambda_h(s)ds.$$

We can now show the following crucial property of $\mathcal{X}_r$.

Lemma 16. *The set $\mathcal{X}_r$ is bounded.*

Proof. Let $x_0 \in \mathcal{X}_r$ be an initial state of the nominal system (3). Denote by $x(t, x_0)$, $t \in [0, T_p]$, the predicted trajectory of

$$\dot{x} = f(x, u), \qquad x(t_0) = x_0$$

for some control trajectory with $u(\tau) \in \mathcal{U}$ and with $x(T_p) = x(T_p, x_0) \in \mathcal{X}_f$. Then,

$$x(t, x_0) - x(T_p) = -\int_t^{T_p} f(x(s, x_0), u(s))ds.$$

Hence, due to the triangle inequality and (4)

$$\begin{aligned}
&\|x(t, x_0) - x(T_p)\| \\
&\quad \leq \int_t^{T_p} \|f(x(s, x_0), u(s))\|ds \\
&\quad \leq \int_t^{T_p} \|f(x(s, x_0), u(s)) - f(x(T_p), u(s))\|ds \\
&\qquad + \int_t^{T_p} \|f(x(T_p), u(s))\|ds \\
&\quad \leq \int_t^{T_p} v\|x(s, x_0) - x(T_p)\|ds + h_0(T_p - t)
\end{aligned}$$

with $h_0 := \|f(x(T_p), u(s))\|$. Application of Lemma 15 to the function $\|x(t, x_0) - x(T_p)\|$ results in

$$\|x(t, x_0) - x(T_p)\| \leq \frac{h_0}{v}\left(e^{vT_p} - e^{vt}\right).$$

Since $x_0 = x(0, x_0)$ and $x(T_p) \in \mathcal{X}_f$, it directly follows

$$\begin{aligned}
\|x_0\| &\leq \|x(T_p)\| + \|x(0, x_0) - x(T_p)\| \\
&\leq \sqrt{\frac{\alpha}{\lambda_{\min}(P)}} + \frac{h_0}{v}(e^{vT_p} - 1).
\end{aligned}$$

Since $x(T_p) \in \mathcal{X}_f$ and $u(s) \in \mathcal{U}$,

$$h := \max_{\substack{u\in\mathcal{U} \\ x\in\mathcal{X}_f}} \|f(x, u)\|$$

is finite and we have

$$\|x_0\| \leq \sqrt{\frac{\alpha}{\lambda_{\min}(P)}} + \frac{h}{v}(e^{vT_p} - 1).$$

This completes the proof.

Furthermore, denote

$$\Delta M := \sup_{x(t_i)\in\mathcal{X}_r} \Delta M(x(t_i)).$$

Since $\mathcal{X}_r$ is a bounded set, the existence of ΔM is guaranteed.

Lemma 17. $\Delta M \to 0$ *for $\beta \to 0$ with $\|w(t)\| \leq \beta$.*

Proof. For any $x(t) \in \mathcal{X}_r$,

$$\begin{aligned}
\Delta M(x(t)) = &\int_t^{t+T_p} \|x_R(t + s, x(t))\|_Q^2 \\
&- \|x(t + s, x(t))\|_Q^2 ds \\
&+ \|x_R(t + T_p, x(t))\|_P^2 - \|x(t + T_p, x(t))\|_P^2.
\end{aligned}$$

Denote

$$\Delta x(t + s, x(t)) := x_R(t + s, x(t)) - x(t + s, x(t)),$$

for all $s \in [0, T_p]$, and

$$q_r := \max_{x\in\mathcal{X}_r} \|x\|,$$

respectively. Then, $\|\Delta x(t + s, x(t))\| \leq \frac{\beta}{v}(e^{vs} - 1)$.

Furthermore,

$$\begin{aligned}&\|x_R(t+s,x(t))\|_Q^2-\|x(t+s,x(t))\|_Q^2\\&\quad=2x(t+s,x(t))^TQ\Delta x(t+s,x(t))+\|x(t+s,x(t))\|_Q^2\\&\quad\le 2\lambda_{\max}(Q)q_r\|\Delta x(t+s,x(t))\|\\&\qquad+\lambda_{\max}(Q)\|\Delta x(t+s,x(t))\|^2\end{aligned}$$

and

$$\begin{aligned}&\|x_R(t+T_p,x(t))\|_P^2-\|x(t+T_p,x(t))\|_P^2\\&\quad=2x(t+T_p,x(t))^TP\Delta x(t+T_p,x(t))\\&\qquad+\|x(t+T_p,x(t))\|_P^2\\&\quad\le 2\lambda_{\max}(P)q_r\|\Delta x(t+T_p,x(t))\|\\&\qquad+\lambda_{\max}(P)\|\Delta x(t+T_p,x(t))\|^2.\end{aligned}$$

That is,

$$\begin{aligned}&\Delta M(x(t))\\&\quad\le\int_t^{t+T_p}2\lambda_{\max}(Q)\|x(t+s,x(t))\|\,\|\Delta x(t+s,x(t))\|\\&\qquad+\lambda_{\max}(Q)\|\Delta x(t+s,x(t))\|^2ds\\&\qquad+2\lambda_{\max}(P)\|x(t+T_p,x(t))\|\,\|\Delta x(t+T_p,x(t))\|\\&\qquad+\lambda_{\max}(P)\|\Delta x(t+T_p,x(t))\|^2.\end{aligned}\tag{9}$$

Integrating the right-hand side of (9) by parts, we obtain

$$\begin{aligned}\Delta M(x(t))\le\frac{\beta}{v}\Bigg(&2\lambda_{\max}(Q)q_r\left(\frac{e^{vT_p-1}}{v}-T_p\right)\\&+\frac{\beta}{v}\left(\frac{e^{2vT_p}-1}{2v}+\frac{2-2e^{vT_p}}{v}+T_p\right)\\&+2\lambda_{\max}(P)q_r\left(\frac{e^{vT_p-1}}{v}-T_p\right)\\&+\frac{\beta}{v}\left(e^{2vT_p}-2e^{vT_p}+1\right)\Bigg).\end{aligned}$$

Let $M_e=2\lambda_{\max}(Q)q_r\left(\frac{e^{vT_p-1}}{v}-T_p\right)+\frac{\beta}{v}\left(e^{2vT_p}-2e^{vT_p}+1\right)+2\lambda_{\max}(P)q_r\left(\frac{e^{vT_p-1}}{v}-T_p\right)+\frac{\beta}{v}\left(\frac{e^{2vT_p}-1}{2v}+\frac{2-2e^{vT_p}}{v}+T_p\right)$, which is finite for a fixed β. Thus, $\Delta M(x(t))\to 0$ for $\beta\to 0$.

Therefore, $\Delta M(x)\to 0$ for $\beta\to 0$ because $\Delta M(x(t))\to 0$ uniformly in $x(t)$.

For $s^2=\frac{\alpha}{\lambda_{\max}(P)}$, we define a set

$$\mathcal{B}_s:=\left\{x\in\mathbb{R}^{n_x}\mid x^Tx\le s^2\right\}.$$

Thus, $\mathcal{B}_s\subseteq\mathcal{X}_f$. Furthermore, $\|x\|^2>s^2$ for all $x\notin\mathcal{B}_s$. The following theorem shows that the state of the real system converges to $\mathcal{B}_s$ as well as $\mathcal{X}_f$.

Theorem 18. *Suppose that*

(a) *the exogenous disturbance satisfies* $\|w(\cdot)\|\le\beta$ *for* $\beta>0$ *small enough such that* $\beta\le\beta_0, s\ge\frac{\beta}{v}(e^{v\delta}-1)$ *and* $\Delta M\le\rho\lambda_{\min}(Q)\delta(s-\frac{\beta}{v}(e^{v\delta}-1))^2$ *with* $\rho\in(0,1)$ *and*
(b) *Problem 4 has a feasible solution at the initial time instant* t_0.

Then,

(1) *Problem 4 is feasible for all* $t\ge t_0$,
(2) *the system state converges asymptotically to the set* $\mathcal{X}_f$, *i.e.,* $\lim_{t\to\infty}d(x_R(t),\mathcal{X}_f)=0$.

Proof. (1) Since $\|w(t)\|\le\beta_0$, recursive feasibility is deduced directly from Theorem 13.

(2) From the definition of $J^0(x)$, it is directly clear that $J^0(0)=0$ and $J^0(x)>0$ for all $x\ne 0$.

For the sake of contradiction, assume the trajectory of the closed loop stays outside of $\mathcal{B}_s$, i.e., $x(t)\notin\mathcal{B}_s$ for all $t\ge t_0$. Then, $x(t_i)\notin\mathcal{B}_s$ for all sampling times t_i.

We know that

$$\begin{aligned}&\bar{J}(x^*(t_i+\delta,x(t_i)))-J^0(x(t_i))\\&\quad\le-\int_{t_i}^{t_i+\delta}\left(\|x^*(\tau,x(t_i))\|_Q^2+\|u^*(\tau,x(t_i))\|_R^2\right)d\tau,\end{aligned}$$

and, consequently,

$$\begin{aligned}&J^0(x_R(t_i+\delta,x(t_i)))-J^0(x(t_i))\\&\quad\le\bar{J}(x_R(t_i+\delta,x(t_i)))-\bar{J}(x^*(t_i+\delta,x(t_i)))\\&\qquad-\int_{t_i}^{t_i+\delta}\left(\|x^*(\tau,x(t_i))\|_Q^2+\|u^*(\tau,x(t_i))\|_R^2\right)d\tau.\end{aligned}\tag{10}$$

Due to $\bar{J}(x_R(t_i+\delta,x(t_i)))-\bar{J}(x^*(t_i+\delta,x(t_i)))\le\Delta M(x(t_i))$, Eq. (10) can be rewritten as

$$\begin{aligned}&J^0(x_R(t_i+\delta,x(t_i)))-J^0(x(t_i))\\&\quad\le\Delta M(x(t_i))\\&\qquad-\lambda_{\min}(Q)\int_{t_i}^{t_i+\delta}\|x^*(\tau,x(t_i))\|^2d\tau\\&\qquad-\lambda_{\min}(R)\int_{t_i}^{t_i+\delta}\|u^*(\tau,x(t_i))\|^2d\tau.\end{aligned}$$

Since $\|x_R(\tau,x(t_i))\|>s$ and $\|x_R(\tau,x(t_i))-x(\tau,x(t_i))\|\le\frac{\beta}{v}(e^{v\tau}-1)$, for all $\tau\in[t_i,t_i+\delta]$,

$$\begin{aligned}\|x(\tau,x(t_i))\|&\ge\Big|\,\|x_R(\tau,x(t_i))\|\\&\qquad-\|x_R(\tau,x(t_i))-x(\tau,x(t_i))\|\,\Big|\\&\ge s-\frac{\beta}{v}(e^{v\tau}-1)\\&\ge s-\frac{\beta}{v}(e^{v\delta}-1).\end{aligned}$$

Thus,

$$\begin{aligned}&J^0(x_R(t_i+\delta,x(t_i)))-J^0(x(t_i))\\&\quad<\Delta M(x(t_i))-\lambda_{\min}(Q)\delta\left(s-\frac{\beta}{v}(e^{v\delta}-1)\right)^2.\end{aligned}$$

Therefore, with $\Delta M(x(t_i))\le\Delta M$ and the assumption $\Delta M\le\rho\lambda_{\min}(Q)\delta(s-\frac{\beta}{v}(e^{v\delta}-1))^2$ in (a)

$$\begin{aligned}&J^0(x_R(t_i+\delta,x(t_i)))-J^0(x(t_i))\\&\quad<(\rho-1)\lambda_{\min}(Q)\left(s-\frac{\beta}{v}(e^{v\delta}-1)\right)^2<0.\end{aligned}\tag{11}$$

By induction, $J^0(x_R(t_i+\delta,x(t_i)))\to-\infty$ for $i\to\infty$ that contradicts, however, the fact that $J^0(x)\ge 0$. Thus, there exists a $t\ge t_0$ such that $x(t)\in\mathcal{B}_s\subset\mathcal{X}_f$.

Since the cost function $J^0(x)$ is monotonically decreasing with respect to bounded disturbances while x lies outside $\mathcal{B}_s$, see Eq. (11), the system trajectory will enter the set $\mathcal{X}_f$ in finite time and will stay in it.

Note that $\Delta M\le\rho\lambda_{\min}(Q)\delta(s-\frac{\beta}{v}(e^{v\delta}-1))^2$ is always satisfied for β small enough because $\Delta M\to 0$ for $\beta\to 0$. Hence, Theorem 18 proves that the system under control is asymptotically ultimately bounded for a small enough disturbance.

Lemma 19. *The optimal cost function $J^0(x)$ is continuous at $x = 0$.*

Proof. In order to show the continuity of $J^0(x)$ at $x = 0$, we now investigate the case that x belongs to some neighborhood of the origin and $x \neq 0$.

Since $x^T Px$ is continuous at $x = 0$, for given $\iota > 0$, there exists $\eta = \eta(\iota) > 0$ such that $x^T Px < \iota$ whenever $\|x\| < \eta$. Define a set

$$\mathcal{X}_\eta := \{x \in \mathbb{R}^{n_x} \mid \|x\| < \eta\}.$$

Without loss of generality, assume that the set $\mathcal{X}_\eta \subset \mathcal{X}_f$. For all $x \in \mathcal{X}_\eta$, a feasible solution to Problem 4, denoted by $\bar{u}(\cdot)$, is

$$\bar{u}(t) = Kx(t, x) \quad \forall t \in [0, T_p]. \tag{12}$$

In accordance with Lemma 7, $J(x, \bar{u}(\cdot)) \le x^T Px$. Since an optimal solution will be no worse than (12), $J^0(0) = 0$ and $J^0(x) > 0$ for all $x \neq 0$, it follows that $\|x\| < \eta$ implies

$$|J^0(x) - J^0(0)| \le J(x, \bar{u}(\cdot)) \le x^T Px < \iota.$$

This implies that $J^0(x)$ is continuous at $x = 0$.

Corollary 20. *Suppose that the disturbance $w(\cdot)$ is decaying, i.e., $\lim_{t\to\infty} \|w(t)\| = 0$, and $\|w(t)\| \le \beta_0$, for all $t \ge t_0$. Then, the system (1) with model predictive control law is asymptotically stable.*

Proof. Recursive feasibility is deduced directly from Theorem 13 because $\|w(t)\| \le \beta_0$.

Denote $\beta(t) := \max_{\tau \ge t} \|w(\tau)\|$, and

$$\Psi_0(x(t)) := \left\{ x \in \mathbb{R}^{n_x} \mid \|x - x^*(t+\delta, x(t))\| \le \frac{\beta(t)}{v}(e^{v\delta} - 1) \right\}.$$

Due to $\|w(\tau)\| \le \beta(t)$ for all $\tau \ge t$, we have $x_R(t+\delta, x(t)) \in \Psi_0(x(t))$.

Denote the supremum and the infimum of $J^0(x)$ in the set $\Psi_0(x(t))$ by M_{Ψ_0} and m_{Ψ_0}, respectively and define

$$\Delta M_{\Psi_0} = M_{\Psi_0} - m_{\Psi_0}.$$

Since $\beta(t) \to 0$ as $t \to \infty$, $\Psi_0(x(t)) \to \{x^*(t+\delta, x(t))\}$ as $t \to \infty$. Thus, $\Delta M_{\Psi_0} \to 0$ as $t \to 0$. That is, for any $\epsilon > 0$, there exists $t_1 > 0$ such that $\Delta M_{\Psi_0} \le \epsilon$ for all $t \ge t_1$.

Similar to the proof of Theorem 18, assume there exists a positive constant ς such that the trajectory of the closed loop stays outside of $\mathcal{B}_\varsigma$, i.e., $x(t) \notin \mathcal{B}_\varsigma$ for all $t \ge t_0$. Then, $x(t_i) \notin \mathcal{B}_\varsigma$ for all sampling times t_i.

In terms of (10), we have

$$\begin{aligned} &J^0(x_R(t_i+\delta, x(t_i))) - J^0(x(t_i)) \\ &\quad \le \epsilon - \lambda_{\min}(Q) \int_{t_i}^{t_i+\delta} \|x^*(\tau, x(t_i))\|^2 d\tau \\ &\qquad - \lambda_{\min}(R) \int_{t_i}^{t_i+\delta} \|u^*(\tau, x(t_i))\|^2 d\tau. \end{aligned}$$

Since

$$\begin{aligned} \lim_{t_i\to\infty} \|x_R(\tau, x(t_i)) - x(\tau, x(t_i))\| &\le \lim_{t_i\to\infty} \frac{\beta(t_i)}{v}(e^{v\tau} - 1) \\ &\to 0, \end{aligned}$$

as $t_i \to \infty$,

$$\|x_R(\tau, x(t_i)) - x(\tau, x(t_i))\| \le \frac{\varsigma}{2}, \quad \forall \tau \in [t_i, t_i+\delta].$$

Furthermore,

$$\begin{aligned} \|x(\tau, x(t_i))\| &\ge \Big| \|x_R(\tau, x(t_i))\| - \|x_R(\tau, x(t_i)) - x(\tau, x(t_i))\| \Big| \\ &\ge \frac{\varsigma}{2}. \end{aligned}$$

Thus,

$$J^0(x_R(t_i+\delta, x(t_i))) - J^0(x(t_i)) \le \epsilon - \frac{\lambda_{\min}(Q)\delta\varsigma^2}{4}.$$

Since ϵ is any positive constant,

$$J^0(x_R(t_i+\delta, x(t_i))) - J^0(x(t_i)) < 0.$$

By induction, $J^0(x_R(t_i+\delta, x(t_i))) \to -\infty$ for $i \to \infty$ that contradicts, however, the fact that $J^0(x) \ge 0$.

Together with $J^0(0) = 0$, $J^0(x) > 0$ for all $x \neq 0$ and $J^0(x)$ being continuous at the origin, the system (1) with model predictive control law is asymptotically stable.

Based on the discussion, we conclude that the degree of inherent robustness of quasi-infinite horizon MPC of nonlinear system with respect to bounded disturbances $w(t)$ depends on

(i) the choice of the terminal set and the terminal penalty function,
(ii) the upper bound on the disturbance $\|w(t)\|$,
(iii) the prediction horizon T_p,
(iv) the Lipschitz constant of the considered system.

Remark 21. The framework in this paper is quite general and, e.g., the following two cases can be treated.

(a) If model perturbations exist: Suppose the nominal system is $\dot{x} = f(x, u)$ and the real system is $\dot{x} = f(x, u) + \Delta f(x, u)$ where the perturbation term $\Delta f(x, u)$ is continuous, then the disturbances w of this paper are $w := \Delta f(x, u)$. The perturbation term $\Delta f(x, u)$ is bounded in the set $\mathcal{X}_r \times \mathcal{U}$ and the bound is independent of x and u.
(b) If the full state were not measured, or if there were some delays: Suppose that the system state is x, and the observed state (or the measured state with delay) is $\bar{x}$, then the disturbances w of this paper is $w := f(x, u) - f(\bar{x}, u)$.

Remark 22. Strictly speaking, a system with model predictive control is a sampled-data control system, where the plant is continuous and the applied input based on the repeated solution of an optimization problem is discrete. The main issue in sampled-data control of nonlinear systems is that for a continuous-time nonlinear system it is in general not possible to derive an exact discrete-time model (Chen & Francis, 1995; Findeisen, 2004; Nesic & Teel, 2001). Thus, for the design of the controller one either designs a continuous-time controller based on a continuous-time plant model, then implements an approximated version of this controller in discrete time. Or one has to find an approximate discrete-time model of the plant then designs a discrete-time controller based on this discrete-time model and finally implements the designed discrete-time controller using a sampler and hold device (Nesic & Teel, 2001). In this paper, the inherent robustness of a continuous-time model predictive control based on a continuous-time plant model is analyzed which also makes the implementation of the approximated version of this controller in discrete time more reliable.

Remark 23. Since the worst-case disturbances for a general class of nonlinear systems with only few assumptions are considered, the estimate of the upper bound of the norm of the admissible disturbances can be expected to be conservative. Future research might focus on a less conservative estimate of the upper bound of the uncertainties and disturbances.

The following section discusses the results presented so far for the special case of linear systems. It will be shown that the properties of linear systems can be used to prove directly that quasi-infinite horizon MPC is input-to-state stable (ISS).

4. Further results on linear systems

In Section 3, we proved that nonlinear systems under quasi-infinite horizon MPC with respect to persistent disturbances converge to the terminal set, and the origin is stable if the disturbances are decaying. The optimal cost function, which is chosen as the candidate Lyapunov function of the quasi-infinite horizon MPC, is possibly a discontinuous function of x. In principle, input-to-state stability property of nonlinear systems can be characterized with a possibly discontinuous Lyapunov function (Grüne, 2002, Theorem 4). However, the ISS property of nonlinear systems under quasi-infinite horizon MPC cannot be asserted directly since the obtained properties of the optimal cost function are only related to the nominal system, but not the real system. It will be shown here that linear systems under quasi-infinite horizon MPC with respect to persistent disturbances are ISS.

Note that nonlinear systems under nominal MPC are ISS if the set of constraints of the optimization problem does not depend on the state, and linear systems under nominal MPC are ISS if the set of constraints is a closed convex polyhedron and the cost function is linear, or if the set of constraints is a compact convex polyhedron and the cost function is continuous (Limon et al., 2009). Under the assumption that the feasible set is a robust positively invariant set for the closed-loop systems, regional input-to-state stability of nominal MPC is achieved for discrete-time nonlinear systems (Magni, Raimondo, & Scattolini, 2006).

Suppose that the considered system is linear, that is, the real system is

$$\dot{x}_R(t) = Ax_R(t) + Bu(t) + w(t),$$

and the nominal system is

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$. Moreover, $x(t)$, $x_R(t)$, and $w(t)$ have the same dimensions and constraints as the ones introduced in (1) and (3).

Similar to Lemma 7, the following lemma shows that the LQR control law Kx, which is a linear quadratic *optimal* control law, can be chosen as the terminal control law.

Lemma 24 (*Chen & Allgöwer, 1998; Chen, O'Reilly, & Ballance, 2003*)**.** *Suppose that the pair (A, B) is stabilizable, and that the Riccati equation*

$$(A + BK)^T P + P(A + BK) = -Q - K^T RK$$

admits a unique solution (P, K), where $P \in \mathbb{R}^{n_x \times n_x}$ is positive definite, and $K \in \mathbb{R}^{n_u \times n_x}$. Then, there exists a constant $\alpha \in (0, \infty)$ specifying a neighborhood of the origin $\mathcal{X}_f := \{x \in \mathbb{R}^{n_x} \mid x^T Px \le \alpha\}$ such that

(1) *$Kx \in \mathcal{U}$, for all $x \in \mathcal{X}_f$, i.e., the linear feedback control law respects the input constraints in the set $\mathcal{X}_f$,*
(2) *$\mathcal{X}_f$ is invariant for the nominal system controlled by the local linear feedback $u = Kx$.*

Denote $E(x) := x^T Px$, Kx and $E(x)$ are the LQR control law and the optimal cost function, respectively, for the related unconstrained linear quadratic optimal control problem for all $x \in \mathcal{X}_f$ (Molinari, 1977). Furthermore, $\frac{dE(x)}{dt} = -x^T(Q + K^T RK)x$ and $E(x(t)) = \int_t^{\infty} x^T(s)(Q + K^T RK)x(s)ds$. Therefore, $E(x)$ can serve as the terminal penalty function of Problem 4.

Notice that for nonlinear systems, the terminal penalty is only an upper bound on the cost to go, but that for linear systems it is an exact equal.

Lemma 25 (*Bemporad, Morari, & Dua, 2002*)**.** *The MPC control law is equivalent to the terminal control law for all $x \in \mathcal{X}_f$.*

Remark 26. $J^0(x)$ is continuously differentiable and unique for all $x \in \mathcal{X}_f$. This is in contrast to the result in Grimm et al. (2004), where the value function of MPC, applied to linear systems with *convex* constraints, is continuous on the interior of the feasible set.

Denote $h^2 = \frac{\alpha}{\lambda_{\max}(P)}$, and define a set

$$\mathcal{B}_h := \left\{x \in \mathbb{R}^{n_x} \mid x^T x \le h^2\right\}.$$

Thus, $\mathcal{B}_h$ is the largest ball inside the terminal set $\mathcal{B}_h$, i.e., $\mathcal{B}_h \subseteq \mathcal{X}_f$.

We can now prove ISS for quasi-infinite horizon MPC applied to linear systems as stated in the following theorem.

Theorem 27. *Suppose that*

(a) *the exogenous disturbance satisfies $\|w(\cdot)\| \le \beta_0$, and $\Delta M_0 \le \rho\lambda_{\min}(Q)\delta h^2$ with $\rho \in (0, 1)$,*
(b) *the optimization problem has a feasible solution at the initial time instant t_0.*

Then,

(1) *the optimization problem is feasible for all $t \ge t_0$,*
(2) *the system state converges to the set $\mathcal{X}_f$,*
(3) *the closed-loop system is input-to-state stable with respect to w.*

Proof. (1) and (2) are deduced directly from Theorems 13 and 18, respectively.

(3) Since $\mathcal{B}_h \subseteq \mathcal{X}_f$, the system trajectory under the MPC control law will also enter into the terminal set $\mathcal{X}_f$ in finite time. Since the optimal cost and the MPC control law are equivalent to the terminal cost and the terminal control law, respectively, we have

$$\lambda_{\min}(P)\|x\|^2 \le J^0(x) \le \lambda_{\max}(P)\|x\|^2, \quad \forall x \in \mathcal{X}_f \tag{13}$$

and

$$\begin{aligned}\frac{d}{dt}J^0(x) &= \frac{d}{dt}x^T Px = \dot{x}^T Px + x^T P\dot{x} \\ &= x^T\left((A + BK)^T P + P(A + BK)\right)x \\ &\quad + w^T Px + x^T Pw.\end{aligned}$$

Since $\frac{dE(x)}{dt} = -x^T(Q + K^T RK)x$ along the nominal trajectory, we know that

$$\begin{aligned}\frac{d}{dt}J^0(x) &= -x^T(Q + K^T RK)x + w^T Px + x^T Pw \\ &\le -\pi_0 x^T Px + w^T Px + x^T Pw \\ &\le -\pi_0 x^T Px + 2\|w\|\,\|P\|\,\|x\|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}J^0(x) \le -\nu\pi_0\lambda_{\min}(P)\|x\|^2, \\ &\qquad \forall\|x\| \ge \frac{2\|P\|}{(1-\nu)\pi_0\lambda_{\min}(P)}\|w\|,\end{aligned} \tag{14}$$

with $\nu \in (0, 1)$. Thus, based on (11), (13) and (14), we conclude that the system is ISS in its feasible region.

Theorem 27 proves the inherent robustness properties of quasi-infinite horizon MPC of constrained linear systems, especially the ISS property of the systems under control. The optimal cost function is an ISS-Lyapunov function in the terminal set $\mathcal{X}_f$.

5. Numerical examples

For demonstrating the proposed analysis scheme, we consider the following two examples.

5.1. Example 1

Consider the system described by

$$\dot{x}(t) = x(t) + 4u(t) + w(t), \tag{15}$$

which is an open-loop unstable linear system with $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $w(t) \in \mathbb{R}$. Assume that x can be measured and the control constraint is

$$-1 \le u(t) \le 1, \quad \forall t \ge 0.$$

The weighting matrices Q and R are chosen as $Q = 1$ and $R = 1$. Solving the LQR problem with the weighting matrices, we get a linear state feedback control matrix $K = -1.2808$. The related Lyapunov matrix is $P = 0.3202$, the terminal set is $\mathcal{X}_f = \{x \in \mathbb{R} \mid x^T P x \le 0.1952\}$. The Lipschitz of the system is $\upsilon = 1$, $\pi_0 = \lambda_{\min}(Q + K^T R K)/\lambda_{\max}(P) = 8.2462$. The open-loop optimization problem described by Problem 4 is solved in discrete time with a sample time of $\delta = 0.1$ time units and a prediction horizon of $T_p = 0.3$ time units. Thus, according to Theorem 13 the nominal MPC is feasible at all time instants if $|w(t)| \le \beta_0 = 0.5364$ for all $t \ge 0$ and the nominal optimization problem is feasible at the initial time instant.

Denote the function $sg(\cdot)$ as

$$sg(\tau) := \begin{cases} 1 & \tau \in [2k\delta, (2k+1)\delta) \\ -1 & \tau \in [(2k+1)\delta, (2k+2)\delta), \end{cases}$$

with $k \in \mathbb{Z}_{[0,\infty)}$. The system trajectory starting from $x(0) = 3$ with nominal MPC and the control input is shown in Fig. 2, where $w(t) \equiv 0.5$, $w(t) \equiv -0.5$ and $w(t) = sg(t) \cdot 0.5$ for all $t \ge 0$, respectively, and J^0 is the optimal cost function. From Fig. 2, we know that the actual system is ultimately bounded under nominal MPC control law for the disturbances, i.e., the system state converges to the terminal set. Compared with the results in Grimm et al. (2004), not only the inherent robustness properties are confirmed, but also an upper bound on the admissible disturbances is estimated.

5.2. Example 2

The system described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + u(t)\,(\vartheta + (1-\vartheta)x_1(t)) + w(t), \\ \dot{x}_2(t) &= x_1(t) + u(t)\,(\vartheta - 4(1-\vartheta)x_2(t)), \end{aligned} \tag{16}$$

with $\vartheta = 0.8$. As pointed out in Chen and Allgöwer (1998), the nominal system of (16) is unstable and stabilizable (but not controllable). Assume that $x_1(t)$ and $x_2(t)$ can be measured, and the control constraint is

$$-2 \le u(t) \le 2, \quad t \ge 0.$$

The weighting matrices are chosen as

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad R = 1. \tag{17}$$

Solving the LQR control problem with these weighting matrices for the locally linearized system, we obtain a linear state feedback gain $\boxed{K = [2.118 \quad 2.118]}$. The Lyapunov matrix $P = \begin{bmatrix} 5.9506 & 0.9506 \\ 0.9506 & 5.9506 \end{bmatrix}$, and the terminal set $\mathcal{X}_f = \{x \in \mathbb{R}^2 \mid x^T P x \le 5.2035\}$ in which the linear control law satisfies the input constraint. Accordingly, $\pi_0 = \lambda_{\min}(Q + K^T R K)/\lambda_{\max}(P) = 0.0725$. The open-loop optimization problem described by Problem 4 is solved in discrete time with a sample time of $\delta = 0.1$ time units and a prediction horizon of $T_p = 1.5$ time units. Since $\left\| \frac{\partial f}{\partial x} \right\| \le \sigma_{\max}$, we use $\sigma_{\max}$ for all $-2 \le u \le 2$ to get an estimate of the Lipschitz. Thus, $\upsilon = 2.0142$. Due to Theorem 13, the optimization problem is feasible for all time instants if $|w(t)| \le \beta_0 = 2.6 \times 10^{-4}$ for all $t \ge 0$ and the nominal optimization problem is feasible at the initial time instant.

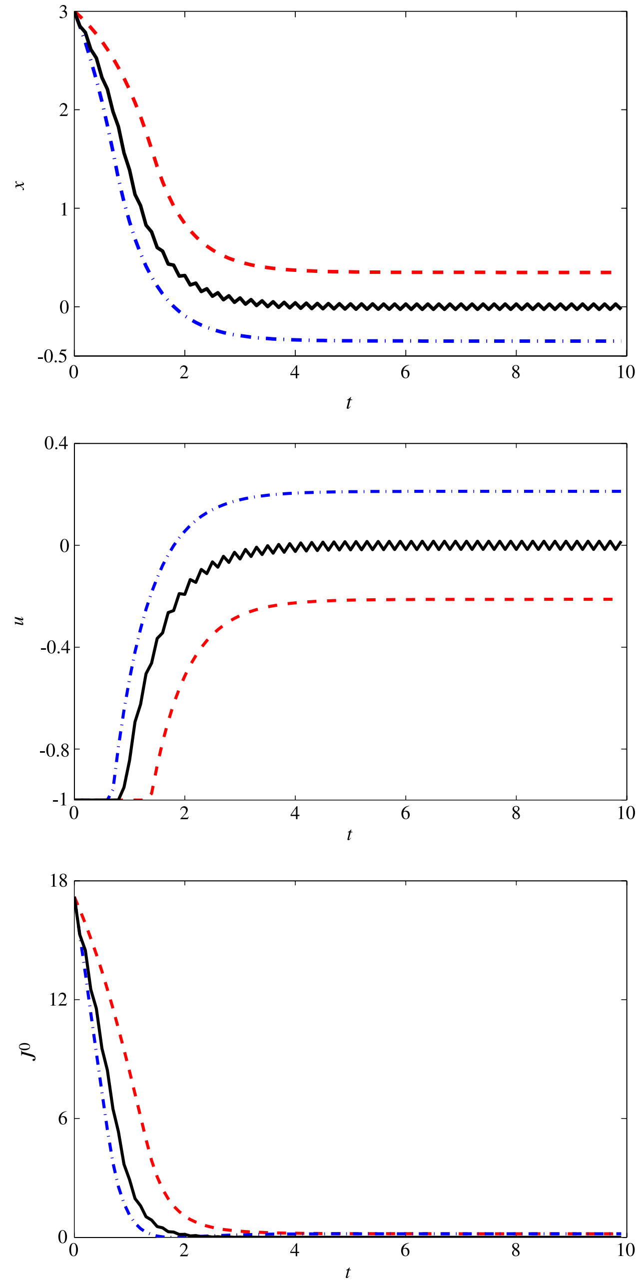

Fig. 2. State and input trajectories and optimal cost function for the initial state $x(0) = 3$. Solid line: case $w(t) = sg(t) \cdot 0.5$, dashed line: case $w(t) \equiv 0.5$, dash–dot line: case $w(t) \equiv -0.5$.

While the initial state of the system (16) is $[0.7\ 0.8]^T$, Problem 4 has a feasible solution. Furthermore, the system state $[0.7\ 0.8]^T$ is driven to the equilibrium since the system is asymptotically stable if the optimization problem is feasible at the initial time instant. It shows by simulation that:

(a) if there exists a disturbance $w(t) \equiv 0.16$, Problem 4 is recursive feasibility and the system state will be driven to a set around the equilibrium;
(b) if the disturbance is $w(t) \equiv 0.17$, Problem 4 is infeasible after some steps. That means the model predictive control scheme fails and there is no control input after that infeasible instant.

The system starts under control from $[0.7\ 0.8]^T$ can endure the disturbances $w(t) \equiv 0.16 > 2.6 \times 10^{-4}$, that is because we have to consider the worst-case disturbances. That makes the scheme of estimate as conservative as min–max MPC.

6. Summary

In this paper, we proved that quasi-infinite horizon MPC of nonlinear systems with input constraints possesses some inherent robustness properties. The disturbances considered are persistent but bounded. Different from existing results, the analysis does not depend on assumptions on convex constraint sets, recursive feasibility, and continuity of the cost function or of the MPC control law. Instead, recursive feasibility was directly proven and robust stability with respect to persistent but bounded disturbances has been addressed by showing that the system trajectory will converge to a set around the origin. The results are of theoretical interest as they indicate that small disturbances can be tolerated when using quasi-infinite horizon MPC.

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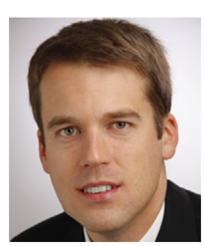

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